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HOMOTOPY PERTURBATION AND ABOODH TRANSFORM FOR SOLVING SINE-GORDEN AND KLEIN-GORDEN EQUATIONS

Mohand M. Abdelrahim Mahgoub*
[*] Mathematics Department Faculty of Sciences-Omderman Islamic University -Khartoum-Sudan
Mathematics Department Faculty of Sciences and Arts-Almikwah -Albaha University- Saudi Arabia

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ABSTRACT
In this paper, we practiced relatively new, analytical method known as the homotopy perturbation method (HPM) and Aboodh transform is employed to obtain the approximate analytical solution of the Klein–Gordon and sine-Gordon equations. The nonlinear terms can be handled by the use of homotopy perturbation method. The proposed homotopy perturbation method is applied to reformulate the first and the second order initial value problems which leads to the solution in terms of transformed variable, and the series solution that can be obtained by making use of the inverse transformation.

KEYWORDS: Homotopy-perturbation method; Aboodh transform; Sine-Gorden equation; Klein-Gorden equation.

INTRODUCTION
Nonlinear phenomena that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics can be modeled by partial differential equations. A broad class of analytical solutions methods and numerical solutions methods were used to handle these problems (Wazwaz 2006). In this paper, we consider the Klein–Gordon equation

$$u_{tt} - u_{xx} + \alpha g(u) = f\ (x,t) \qquad (1)$$

and

$$u_{tt} - u_{xx} + \beta_1 u + \beta_2 g(u) = f\ (x,t) \qquad (2)$$

where u is a function of x, t and g is a nonlinear function . The α parameter is so-called dissipative term, which is assumed to be a real number with $\alpha \geq 0$. When $\alpha = 0$, Eq.(1), reduces to the undamped SG equation, and when $\alpha > 0$, to the damped one. f is also a known analytic function.

In Quantum Mechanics Klein Gordon mathematical model is one of the most important model. also find its application in condensed matter physics , in collision Plasma for interaction of solution, initial state recurrence and nonlinear wave equation. Many researchers applied different methods to obtain the solution of Klein Gordon equation arises in different fields. Among these methods Taylor method, Modified Adomian decomposition method, Differential Transform method and Pade, Homotopy Analysis Method ,Variation iterational method and Homotopy Perturbation Method [5-9] . By applying these methods we get the solution in series form, which is assumed to be converges to exact solution. In recent years, HPM has been applied to various kinds of nonlinear problems and show great success.

Recently, Khalid Aboodh [1-4], has introduced a new integral transform, named the Aboodh transform, and it has further applied to the solution of ordinary and partial differential equations. Now, we consider in this work the effectiveness of the homotopy-perturbation Aboodh transform method to obtain the exact and approximate analytical solution of the Sine-Gorden and the Klein-Gorden equations[10-12].

Aboodh Transform:

A new transform called the Aboodh transform defined for function of exponential order we consider functions in the set A, defined by:

$$A = \{f(t): \exists\, M, k_1, k_2 > 0, |f(t)| < Me^{-vt}\} \quad (3)$$

For a given function in the set M must be finite number, k_1, k_2 may be finite or infinite. Aboodh transform which is defined by the integral equation

$$A[f(t)] = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}\,dt t \geq 0, k_1 \leq v \leq k_2 \quad (4)$$

Aboodh transform of some partial derivative:

$$A\left[\frac{\partial u(x,t)}{\partial x}\right] = K'(x,v) \quad (5)$$

$$A\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = K''(x,v) \quad (6)$$

$$A\left[\frac{\partial^n u(x,t)}{\partial x^n}\right] = K^{(n)}(x,v) \quad (7)$$

$$A\left[\frac{\partial u(x,t)}{\partial t}\right] = v\,K(x,v) - \frac{u(x,0)}{v} \quad (8)$$

$$A\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = v^2 K(x,v) - \frac{\frac{\partial u(x,0)}{\partial t}}{v} - u(x,0) \quad (9)$$

Therefore, one can easily extend this result to the *nth* partial derivative by using mathematical induction. We will see that, the Aboodh transform may be used to solve intricate problems in engineering, mathematics and applied science without resorting to a new frequency domain.

Homotopy Perturbation Method:

To introduce HPM, considered the following general nonlinear differential equation:

$$Lu + Nu = f\,(x,t) \quad (10)$$

with initial conditions:

$$u(x,0) = k_1 \quad , \quad u_t(x,0) = k_2 \quad (11)$$

Where u is a function of x, t and k_1, k_2 are constants or functions of x. Also L and N are the linear and nonlinear operators respectively. According to HPM [2] we construct a homotopy which satisfies the following relation:

$$H(u,p) = Lu - Lv_0 + p[Lv_0 + Nu - f(x,t)] = 0 \quad (12)$$

where $p \in [0,1]$ is an embedded parameter and v_0 is an arbitrary initial approximation satisfying the given initial condition.

By setting $p = 0$ and $p = 1$ in Eq.(12), one obtain:

$$H(u,0) = Lu - Lv_0 = 0 \text{ and } H(u,1) = Lu + Nu - f\,(x,t) = 0 \quad (13)$$

which are the linear and nonlinear original equations, respectively. In topology, this is called deformation and $Lu - Lv_0$ and $Lu + Nu - f\,(x,t)$ are called homotopic. Here, the embedded parameter is introduced much more naturally, unaffected by artificial factor, further it can be considered as a small parameter for $0 \leq p \leq 1$.

Chowdhury and Hashim in [2], have presented an alternative way for choosing the initial approximation, that is:

$$v_0 = u(x,0) + tu_t(x,0) + L^{-1}f\,(x,t) = k_1 + tk_2 + L^{-1}f\,(x,t), \quad (14)$$

where $L^{-1} = \int_0^t \int_0^t \dots \int_0^t dtdt \dots dt$ depends on the order of the linear operator. With this assumption that the initial approximation v_0 given in Eq.(14), in HPM, the solution of Eq.(13), is expressed as:

$$u(x,t) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + \dots \quad (15)$$

Hence the approximate solution of Eq.(10), can be expressed as a power series of p, i. e.

$$u = \lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i \quad (16)$$

Homotopy Perturbation and Aboodh Transform Method:

Consider a general nonlinear partial differential equation with initial conditions of the form:

$$Du(x,t) + Ru(x,t) + Nu(x,t) = f(x,t) \quad (17)$$

with the following initial conditions

$$u(x,0) = c_1,\ u_t(x,0) = c_2 \quad (18)$$

Where D is a linear differential operator of order two, R is a linear differential operator of less order than D, N is the general nonlinear differential operator ,$f(x,t)$ is the source term and c_1 ,c_2 are constants or functions of x.

By taking Aboodh transform to both sides of Eq.(17), result in:

$$A[Du(x,t)] + A[Ru(x,t)] + A\,[Nu(x,t)] = A\,[f(x,t)] \qquad (19)$$

Using the differentiation property of Aboodh transform and the initial condition in Eq.(18)one obtain:

$$A[u(x,t)] = \frac{1}{v^2}A[f(x,t)] + \frac{1}{v^2}c_1 + \frac{1}{v^3}c_1 - \frac{1}{v^2}A[Ru(x,t) + Nu(x,t)] \qquad (20)$$

Applying the inverse Aboodh transform on both sides of Eq.(20), we get:

$$u(x,t) = F(x,t) - A^{-1}[\frac{1}{v^2}A[Ru(x,t) + Nu(x,t)]] \qquad (21)$$

where $F(x,t)$ represent the term arising from the source term and the prescribed initial conditions. According to HPM we have:

$$u(x,t) = F(x,t) - pA^{-1}[\frac{1}{v^2}A[Ru(x,t) + Nu(x,t)]] \qquad (22)$$

now by substituting

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \;\; , Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \qquad (23)$$

in Eq.(20) where $H_n(u)$ is He's polynomials that are given by:

$$H_n(u_0, u_1, u_2 \dots u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[N(\sum_{=0}^{\infty} p^i u_i)]_{p=0} \;, n = 0,1,2,\dots \qquad (24)$$

one obtain:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) =$$

$$F(x,t) - pA^{-1}[\frac{1}{v2}A[R\sum_{n=0}^{\infty} p^n u_n(x,t) + (N\sum_{n=0}^{\infty} p^n H_n(u))]] \qquad (25)$$

This our method is infact a coupling technique of Aboodh transform and the homotopy perturbation method. Comparing the coefficients of the like powers of p the following approximations are resulted:

$$p^0: \; u_0(x,t) = F(x,t),$$

$$p^1: \; u_1(x,t) = -A^{-1}\left[\frac{1}{v^2}A[Ru_0(x,t) + H_0(u)]\right],$$

$$p^2: \; u_2(x,t) = -A^{-1}\left[\frac{1}{v^2}A[Ru_1(x,t) + H_1(u)]\right],$$

$$p^3: \; u_3(x,t) = -A^{-1}\left[\frac{1}{v^2}A[Ru_2(x,t) + H_2(u)]\right],$$

$$\vdots$$

Then the solution is;

$$u(x,t) = \lim_{p\to 1} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \qquad (26)$$

Applications

Example 5.1

Consider the following Sine-Gorden equation with the given initial conditions:

$$u_{tt} - u_{xx} + \sin u = 0 \;\; , u(x,0) = 0, \; u_t(x,0) = 4\,\mathrm{sech}(x) \qquad (27)$$

The exact solution is given as:

$$u(x,t) = 4\arctan[t\,\sec h(x)\,] \qquad (28)$$

To solve the example by this method, we take $\sin u \cong u - \frac{u^3}{3!} + \frac{u^5}{5!}$

After taking Aboodh transform of Eq.(27), subjected to the initial conditions, one obtain

$$A[u(x,t)] = \frac{4}{v^3}\mathrm{sech}(x) + \frac{1}{v^2}\frac{d^2}{dx^2}A[u(x,t)] - \frac{1}{v^2}A\left[u - \frac{u^3}{3!} + \frac{u^5}{5!}\right] \qquad (29)$$

The inverse Aboodh transform implies that

$$u(x,t) = 4t\,\mathrm{sech}(x) + A^{-1}\left[\frac{1}{v^2}\frac{d^2}{dx^2}A[u(x,t)] - \frac{1}{v^2}A\left[u - \frac{u^3}{3!} + \frac{u^5}{5!}\right]\right] \qquad (30)$$

Now applying the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 4t\,\mathrm{sech}(x) +$$

$$pA^{-1}\left[\frac{1}{v^2}\frac{d^2}{dx^2}A[\sum_{n=0}^{\infty}p^n u_n(x,t)] - \frac{1}{v^2}A\left[\sum_{n=0}^{\infty}p^n u_n(x,t) - \frac{(\sum_{n=0}^{\infty}p^n u_n(x,t))^3}{3!} + \frac{(\sum_{n=0}^{\infty}p^n u_n(x,t))^5}{5!}\right]\right]$$
(31)

By comparing the coefficients of the same powers of p, result in:

$$p^0: u_0(x,t) = 4t\,\mathrm{sech}(x)$$

$$p^1: u_1(x,t) = \frac{d^2}{dx^2}A^{-1}\left[4t\,\mathrm{sech}(x)\frac{1}{v^2}A(t)\right] - A^{-1}\left[4t\,\mathrm{sech}(x)\frac{1}{v^2}A(t)\right]$$
$$+A^{-1}\left[\frac{64}{6}\mathrm{sech}^3(x)\frac{1}{v^2}A(t^3)\right] - A^{-1}\left[\frac{1024}{120}\mathrm{sech}^5(x)\frac{1}{v^2}A(t^5)\right]$$
$$= \frac{4}{315}\mathrm{sech}^5(x) - (-105t^3\cosh^2(x) + 24t^5\cosh^2(x) - 16t^7)$$

$$p^2: u_2(x,t) = \frac{4}{2027025}t^5\mathrm{sech}^9(x)\,[7040t^8 - 33696t^9\cosh^2(x) - 4290t^4\cosh^4(x)$$
$$+ 143000t^4\cosh^2(x) - 205920t^2\cosh^4(x)$$
$$+51480t^2\cosh^6(x) - 270270\cosh^6(x) + 405405\cosh^4(x)]$$

Therefore , the 3-terms Aboodh - HPM solution is

$u(x,t) =$

$$\frac{4}{2027025}t\mathrm{sech}^9(x)\,[7040t^{12} - 33696t^{10}\cosh^2(x) - 4290t^8\cosh^4(x)$$
$$+143000t^8\cosh^2(x) - 308880t^6\cosh^4(x) + 405405t^2\cosh^4(x)$$
$$-675675t^2\cosh^6(x) - 270270\cosh^8(x)$$

Example 5.2

Consider the following Sine-Gorden equation with the given initial conditions:

$$u_{tt} - u_{xx} + \sin u = 0 \;\; , u(x,0) = \pi + \varepsilon\cos(\mu x),\; u_t(x,0) = 0 \qquad (31)$$

Where $\mu = \frac{1}{\sqrt{2}}$ and ε is a constant .

We take $\sin u \cong u - \frac{u^3}{3!} + \frac{u^5}{5!}$ to solve this example .

By taking Aboodh transform of Eq.(31), subjected to the initial conditions, one obtain

$$A[u(x,t)] = \frac{(\pi+\varepsilon\cos(\mu x))}{v^2} + \frac{1}{v^2}\frac{d^2}{dx^2}A[u(x,t)] - \frac{1}{v^2}A\left[u - \frac{u^3}{3!} + \frac{u^5}{5!}\right] \qquad (32)$$

The inverse Aboodh transform implies that

$$u(x,t) = (\pi + \varepsilon\cos(\mu x)) + A^{-1}\left[\frac{1}{v^2}\frac{d^2}{dx^2}A[u(x,t)] - \frac{1}{v^2}A\left[u - \frac{u^3}{3!} + \frac{u^5}{5!}\right]\right] \qquad (33)$$

Now applying the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty}p^n u_n(x,t) = (\pi + \varepsilon\cos(\mu x)) +$$
$$pA^{-1}\left[\frac{1}{v^2}\frac{d^2}{dx^2}A[\sum_{n=0}^{\infty}p^n u_n(x,t)] - \frac{1}{v^2}A\left[\sum_{n=0}^{\infty}p^n u_n(x,t) - \frac{(\sum_{n=0}^{\infty}p^n u_n(x,t))^3}{3!} + \frac{(\sum_{n=0}^{\infty}p^n u_n(x,t))^5}{5!}\right]\right]$$
(34)

By comparing the coefficients of the same powers of p, one obtain :

$$p^0: u_0(x,t) = \pi + \varepsilon\cos(\mu x)$$

$$p^1: u_1(x,t) = \frac{t^2}{2}\left[-\mu^2\varepsilon\cos(\mu x) + \frac{(\pi+\varepsilon\cos(\mu x))^3}{6} - \frac{(\pi+\varepsilon\cos(\mu x))^5}{120} - (\pi + \varepsilon\cos(\mu x))\right]$$
$$\vdots$$

Therefore , the 3-terms Aboodh - HPM solution is

$$u(x,t) = \pi + \varepsilon\cos(\mu x) + \frac{t^2}{2}\left[-\mu^2\varepsilon\cos(\mu x) + \frac{(\pi+\varepsilon\cos(\mu x))^3}{6} - \frac{(\pi+\varepsilon\cos(\mu x))^5}{120} - (\pi + \varepsilon\cos(\mu x))\right] +$$
$$\frac{1}{69120}\Big[\varepsilon^2\cos^2(\mu x) + 9\pi\varepsilon^8\cos^8(\mu x) + (36\pi^2 - 32)\varepsilon^7\cos^7(\mu x) + (84\pi^3 - 224\pi)\varepsilon^6\cos^6(\mu x) +$$
$$720\left(\mu^2 + \frac{7}{40}\pi^4 - \frac{14}{15}\pi^2 + \frac{8}{15}\right)\varepsilon^5\cos^5(\mu x) + 2400\pi\left(\frac{21}{400}\pi^4 + \frac{4}{5} + \mu^2 - \frac{7}{15}\pi^2\right)\varepsilon^4\cos^4(\mu x) -$$
$$480\left((\varepsilon\mu)^2 + (-6\pi^2 + 12)\mu^2 - 8\pi^2 + \frac{7}{3}\pi^4 - \frac{7}{40}\pi^2 + \varepsilon\right)\varepsilon^3\cos^3(\mu x) - 1440\pi\Big((\varepsilon\mu)^2 +$$
$$(-\pi^2 + 6)\mu^2 + \frac{7}{15}\pi^4 - \frac{1}{40}\pi^6 - \frac{8}{3}\pi^2 + \varepsilon\Big)\varepsilon^2\cos^2(\mu x) - 1440\Big((\varepsilon\mu)^2(\pi^2 - 2) - 2\mu^4 + \left\{\frac{-1}{6}\pi^4 -\right.$$
$$\left.4 + 2\pi^2\right\}\mu^{24} + \frac{7}{45}\pi^6 + 4\pi^2 - \frac{1}{160}\pi^8\Big) - \frac{4}{3}\pi^4 - 2)\varepsilon\cos(\mu x) - 480\pi(\pi^2 - 6)(\varepsilon\mu)^2 + 384\pi^5 +$$

$2880\pi - 1920\pi^3 + \pi^9 - 32\pi^7\Big] t^4.$

Example 5.3

Consider the following Klein-Gorden equation with the given initial conditions:

$$u_{tt} - u_{xx} = u \ , \ u(x,0) = 1 + \sin x \, , \ u_t(x,0) = 0 \qquad (35)$$

By taking Aboodh transform of Eq.(35), subjected to the initial conditions, one obtain

$$A[u(x,t)] = \frac{(1+\sin x)}{v^2} + \frac{1}{v^2}\frac{d^2}{dx^2}A[u(x,t)] + \frac{1}{v^2}A[u(x,t)] \qquad (36)$$

The inverse Aboodh transform implies that

$$u(x,t) = (1 + \sin x) + A^{-1}\left[\frac{1}{v^2}\frac{d^2}{dx^2}A[u(x,t)] + \frac{1}{v^2}A[u(x,t)]\right] \qquad (37)$$

Now applying the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = (1 + \sin x) + pA^{-1}\left[\frac{1}{v^2}\frac{d^2}{dx^2}A[\textstyle\sum_{n=0}^{\infty} p^n u_n(x,t)] + \frac{1}{v^2}A[\sum_{n=0}^{\infty} p^n u_n(x,t)]\right] \qquad (38)$$

By comparing the coefficients of the same powers of p, one obtain :

$$p^0 : u_0(x,t) = 1 + sin\, x$$

$$p^1 : u_1(x,t) = \frac{t^2}{2}$$

$$p^2 : u_2(x,t) = \frac{t^4}{24}$$

$$p^3 : u_3(x,t) = \frac{t^6}{720}$$

Therefore , the 4-terms approximate series solution is :

$$u(x,t) = 1 + \sin x + \frac{t^2}{2} + \frac{t^4}{24} + \frac{t^6}{720}$$

and this will, in the limit of infinitely many terms, yield the closed form solution [6],

$$u(x,t) = \sin x + \cos ht$$

Example 5.4

Consider the following Klein-Gorden equation with the given initial conditions:

$$u_{tt} - u_{xx} = -u \ , \ u(x,0) = 0, \ u_t(x,0) = x \qquad (39)$$

By taking Aboodh transform of Eq.(39), subjected to the initial conditions, one obtain

$$A[u(x,t)] = \frac{x}{v^3} + \frac{1}{v^2}\frac{d^2}{dx^2}A[u(x,t)] - \frac{1}{v^2}A[u(x,t)] \qquad (40)$$

The inverse Aboodh transform implies that

$$u(x,t) = tx + A^{-1}\left[\frac{1}{v^2}\frac{d^2}{dx^2}A[u(x,t)] - \frac{1}{v^2}A[u(x,t)]\right] \qquad (41)$$

Now applying the homotopy perturbation method, we get:

$$\textstyle\sum_{n=0}^{\infty} p^n u_n(x,t) = tx + pA^{-1}\left[\frac{1}{v^2}\frac{d^2}{dx^2}A[\sum_{n=0}^{\infty} p^n u_n(x,t)] - \frac{1}{v^2}A[\sum_{n=0}^{\infty} p^n u_n(x,t)]\right] \qquad (42)$$

By comparing the coefficients of the same powers of p, one obtain :

$$p^0 : u_0(x,t) = tx$$

$$p^1 : u_1(x,t) = \frac{-xt^3}{3!}$$

$$p^2: u_2(x,t) = \frac{xt^5}{5!}$$
$$p^3: u_3(x,t) = \frac{-xt^7}{7!}$$

Therefore , the 4-terms approximate series solution is :

$$u(x,t) = tx - \frac{xt^3}{3!} + \frac{xt^5}{5!} - \frac{xt^7}{7!}$$

and this will, in the limit of infinitely many terms, yield the closed form solution [22],

$$u(x,t) = x \sin t$$

CONCLUSION

In this paper, Aboodh Homotopy-Perturbation method (ATHPM) has been successfully applied to solve the Klein–Gordon and sine-Gordon equations. The method is reliable and easy to use. The results show that the ATHPM is a powerful and efficient technique in finding exact and approximate solutions of nonlinear differential equations. By using this method we obtain a new, efficient recurrent relation, to solve to solve the Klein–Gordon and sine-Gordon equations, this means that ATHPM provide highly accurate numerical solutions for nonlinear problems in comparison with other method.

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